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A SURVEY OF RECENT MATHEMATICAL PUBLICATIONS ON THE SUBJECT OF CURVE CROSSINGS BY STOCHASTIC PROCESSES

by

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SUMMARY

This report is prompted by the rash of good mathematical research on the problem of the moments of the number of crossings of a curve by a stochastic process. It is felt that this problem is important to some engineers and designers who may find a brief introduction to these recent publications useful.

It should be understood that this document is not motivated by any assumed ability of the author to judge important research from research of lesser importance; and any slights must be considered accordingly.

Section 1

S. O. Rice, Mathematical analysis of random noise, *Bell System Technical Journal*, Vols. 23 and 24, 1945.

N. Donald Ylvisaker, The expected number of zeroes of a stationary Gaussian process, *Ann. Math. Statist.*, Vol. 36, pp. 1043-1046, 1965.

Rice's paper is mentioned in that the formula for the expected number of crossings of a fixed level by a stationary Gaussian process is often referred to as Rice's formula, in view of his exposition in the *Bell System Journal.*

Since Rice's work there has been much attention given to the problem of determining the minimal hypothesis under which this formula can be rigorously shown to apply. The complete answer was given by Ylvisaker.

Ylvisaker proves that if $X(t)$ is a continuous stationary Gaussian process with mean function zero, then Rice's formula for the expected number of crossings of a certain level a in a given time interval always holds, in the sense that if $\rho''(0)$ does not exist, or is not continuous, then the expected number of crossings is $+\infty$.

Unlike Rice [8] and Kac [2], Ylvisaker uses a method which does not count tangential a-points. It is then necessary to prove that with probability 1 there are no tangential a-points. This is accomplished by first showing that the random variable $\sup_{a<\tau<b} X(\tau)$ has a continuous distribution, hence it is equal to some fixed point a with probability 0. That there can be no tangential (from below) a points follow from the observation that

$$\Pr\{X(t) \text{ has a tangential (from below) a-point}\} \leq$$

$$\Sigma_{j,n} P\{\sup_{t_j-\varepsilon_n<\tau<t_j+\varepsilon_n} X(\tau) = a\}$$

where t_j are dense in the time interval under consideration and $\varepsilon_n \to 0$. The last sum is zero as we have remarked earlier.

To find the expected number of a-points in the time interval $(0,t)$ we define the auxiliary variables

$$U_{n_k} = 1 \quad \text{if} \quad (X((k-1)2^{-n}t)-a)(X(k2^{-n}t)-a) < 0$$

$$= 0 \quad \text{otherwise},$$

$$k=1,2,\ldots,2^n.$$

Then $Z_n = \sum_1^{2^n} U_{n_k}$ is the number of a-crossings by the discrete time process $X(k) = X(k2^{-n}t)$. Z_n is a nondecreasing sequence and as would be expected its limit Z is the number of a-crossings of the $X(\tau)$ process.

Since the U_{n_k} are identically distributed,

$$EZ = L_n(2^n EU_{n_k});$$

so all that remains is to find EU_{n_k}.

By the definition of U_{n_k},

$$EU_{n_k} = \Pr\{[X((k-1)2^{-n}t)-a]\cdot[X(k2^{-n}t)-a] < 0\}.$$

For simiplicity let $Y = X((k-1)2^{-n}t)$, $Z = X(k2^{-n}t)$, then

$$EU_{n_k} = \Pr\{Y<a, Z>a\} + \Pr\{Y>a, Z<a\}$$

which may be rewritten

$$EU_{n_k} = 2[\Pr\{Y>a\} - \Pr\{Y>a,Z>a\}].$$

If we let $I(a,a,\rho(2^{-n}))$ denote the probability that the bivariate normal random variables Y and Z which have correlation $\rho(2^{-n})$ are both greater than a, then

$$EU_{n_k} = 2[L(a,a,\rho(0)) - L(a,a,\rho(2^{-n}))].$$

Hence

$$EZ_n = 2\left[\frac{L(a,a,\rho(0))-L(a,a,\rho(2^{-n}))}{2^{-n}}\right].$$

Using an integral expression for $L(\cdot,\cdot,\cdot)$ (formula (3), page vi, [11])

$$EZ_n = 1/\pi \left\{ \int_{\text{arc cos}(\rho(0))}^{\text{arc cos}(\rho(2^{-n}))} \exp[-a^2(1 + \cos\omega)\operatorname{cosec}^2\omega]d\omega. \right.$$

Over the range of the integral the integrand converges uniformly to $\exp - a^2/2$, hence

$$\operatorname{Lim}_n EZ_n = \operatorname{Lim}_n \frac{e^{-a^2/2}}{\pi} 2^n \text{ arc cos } (\rho\cdot(2^{-n})).$$

Using L'Hospital's rule,

$$\operatorname*{Lim}_{t\to0^+} \frac{\text{arc cos } \rho(t)}{t} = \operatorname*{Lim}_{t\to0^+} \left| \frac{\rho'(t)}{(1-\rho^2(t))^{1/2}} \right|$$

whenever the latter limit exists in the sense that the lim sup is equal to the lim inf.

Squaring, we have:

$$\lim_{t\to 0^+} \frac{[\rho'(t)]^2}{1-\rho^2(t)} = \lim_{t\to 0^+} \frac{-\rho''(t)}{\rho(t)}$$

whenever the latter exists in this generalized sense.

We now use the spectral form to show that in the generalized sense $\lim_{t\to 0} \rho''(t)$ always exists.

Since

$$\rho(t) = \int_0^\infty \cos 2\pi\lambda t \, dG(\lambda),$$

and we can write

$$\lim_{t\to 0} \rho''(t) = \lim_{t\to 0} \int_0^\infty \frac{d^2}{dt^2} [\cos 2\pi\lambda t] dG(\lambda)$$

whenever the integral is absolutely convergent. If $\int_0^\infty \lambda^2 dG(\lambda) < +\infty$, $\rho''(t)$ exists, is continuous and finite. If on the other hand, $\int_0^\infty \lambda^2 dG(\lambda) = +\infty$, then it can be seen from the spectral form that

$$\lim_{t\to 0} \rho''(t) = +\infty.$$

Hence

$$\lim_{t\to 0} \frac{\arccos \rho(t)}{t} = [-\rho''(0)]^{\frac{1}{2}}$$

in the generalized sense.

Section 2

M. R. Leadbetter, On crossings of arbitrary curves by certain Gaussian processes, *Proc. Amer. Math. Soc.*, Vol. 16, No. 1, pp. 60-68, 1965.

M. R. Leadbetter and J. D. Cryer, On the mean number of curve crossings by non-stationary normal processes, *Ann. Math. Statist.*, Vol. 36, pp. 509-516, 1965.

The problem of considering curve crossings instead of crossings of a fixed level is only moderately more difficult. In [3] and [6], Leadbetter and Leadbetter and Cryer attack the problem with the methods originated by Kac; they use a method for counting zeroes which involves the sample derivative. The theory is much the same, but in considering crossings of a curve there is somewhat more bookkeeping. The formula for the expected number of crossings in $(0,T)$ is the integral of the expected number of crossings in dt. Once we have gotten away from the simplicity of the fixed level it is natural to avoid the assumption of stationarity and keep track of the process behavior as well.

The paper [3] considers a separable stationary Gaussian process whose spectral density satisfies

$$\int_0^\infty \lambda^2 [\ell g(1+\lambda)]^b dF(\lambda) < \infty \quad \text{some} \quad b > 1.$$

It follows that the process has a.s. a sample derivative. In addition, it is assumed that the curve $U(t)$ has a continuous first derivative.

The paper [6] by Leadbetter and Cryer goes beyond the above paper in that it avoids the assumption of stationarity. Essentially the requirements are the same: the curve $U(t)$ less the mean function $M(t)$ must have a continuous derivative. The process must have a.s. a continuous sample derivative, and in addition it is assumed that the covariance function $\Gamma(t,t') = \text{cov}\,\{X(t),X(t')\}$ has a mixed second partial derivative which is continuous at all diagonal points (t,t). It follows that $\Gamma_{12}(t,t')$ is everywhere continuous. (For a proof of this fact as well as many related properties of covariance functions see the chapter on "Second Order Properties" in Loève [7].)

Leadbetter and Cryer mention that the existence of the sample derivative is important only to insure that there be no tangential crossings. It is stated that if we consider only actual crossings of the curve the main theorem of [6] follows without the requirement of a derivative.

Indications are that with probability 1 there will be no tangential crossings of a differentiable curve by a continuous Gaussian process, but at this point this has not been proven directly. That there will almost surely be no tangential crossings follows from the fact that the main theorem of [4] reduces to the form of Leadbetter and Cryer in the case that they consider. (The formula of Leadbetter and Cryer counts tangential crossings, the formula of [4] does not.)

Section 3

Harold Cramér and M. R. Leadbetter, The moments of the number of crossings of a level by a stationary normal process, *Ann. Math. Statist.*, Vol. 36, pp. 1656-1663, 1965.

N. Donald Ylvisaker, On a theorem of Cramér and Leadbetter, *Ann. Math. Statist.*, Vol. 37, pp. 681-685, 1966.

M. R. Leadbetter, A note on the number of axis crossings by a stochastic process, *Bull. Amer. Math. Soc.*, Vol. 73, pp. 129-132, January 1967.

These papers detail proofs in increasing generality of a formula for the factorial moments of the number of upcrossings of a curve by a stochastic process.

If N is the number of upcrossings, then the k^{th} factorial moment of N is

$$EN(N-1)\cdots(N-k+1).$$

The moments of the number of downcrossings may be obtained in the same manner, and with a bit more work one can obtain the moments of the total number of crossings.

The reason for considering factorial moments becomes clear if we recall the method of Section 1. There

$$\begin{aligned} U_{n_k} &= 1 \quad \text{if} \quad (X((k-1)2^{-n}t-a)(X(k2^{-n}t)-a) < 0 \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and we concerned ourselves with finding EZ_n, where

$$Z_n = \sum_1^{2^k} U_{n_k}.$$

For the higher moments, consider

$$Z_{n,2} = \sum_{I'} U_{n_j} \cdot U_{n_k},$$

where I' is the collection of all (j,k) such that $j \neq k$, $1 \leq j \leq 2^n$, $1 \leq k \leq 2^n$. Then, if our polygonal version of the $X(t)$ has N crossings,

$Z_{n,2}$ will be equal to the number of ways that a subset of size 2 can be picked from a set of size N, hence

$$Z_{n,2} = N(N-1).$$

Taking expectations:

$$EZ_{n,2} = EN(N-1).$$

Similarly, if

$$Z_{n,k} = \sum_{I''} U_{n_1} \cdots U_{n_k}$$

where I'' is the collection of all index vectors $(n_1,\ldots,n_k)$ having different components, then

$$EZ_{n,k} = EN(N-1) \cdots (N-k+1).$$

As in Section 1, the problem is then to find the limits:

$$\lim_{n\to\infty} 2^{kn} \Pr\{U_{n_1} \cdots U_{n_k} = 1\},$$

and to show that the limit of the integral is in fact the integral of the limit.

In the Cramér-Leadbetter paper it is shown that

i) $$E_T N(N-1) \cdots (N-k+1) = \int_{\mathbf{T}} dt \int_{\mathbf{P}} \left(\prod_1^k y_i\right) p_t(a,y)\,dy$$

where $\mathbf{T}$ is the k-dimensional cube $[0,T] \times \cdots \times [0,T]$, $\mathbf{P}$ is the k-dimensional cube $[0,+\infty] \times \cdots \times [0,+\infty]$, and $p_t(a,y)$ is the joint density function of $(X'(t_1),\ldots,X'(t_k),X(t_1),\ldots,X(t_k))$ and a is the vector k-dimensional vector $(a,\ldots,a)$.

It should be pointed out that neither the considerations leading to the expression for $Z_{n,k}$ nor the resulting formula of Cramér-Leadbetter depend on normality or stationarity. In fact if the vector a in i) is replaced by $a(t)$, the formula i) is meaningful for continuous curves $a(t)$. All that has been required is that the process $X(t)$ have some sort of sample derivative $X'(t)$. In Ylvisaker's first paper [9] it was shown that if $X'(t)$ did not almost surely exist in a mean square sense, then $EN = +\infty$; and it follows that $EN(N-1) \cdots (N-k+1) = +\infty$. In the paper [10] Ylvisaker completes the paper of Cramér-Leadbetter in the manner that he treated Rice's formula; it is shown that the Cramér-Leadbetter formula holds for any stationary normal process in the sense that if $X'(t)$ does not exist then both sides of i) are the limit of sequences to which converge to $+\infty$. Ylvisaker utilizes the limit theorems of Martingale theory in a way which makes the detailed analysis of passing to the limit quite pleasant.

In [5], "A note on the number of axis crossings" by M. R. Leadbetter, it is shown that a formula analogous to that of Cramér-Leadbetter is valid for processes and curves satisfying a minimal of regularity conditions.

It is impossible to evaluate the limits necessary within the broad framework of non-normal processes, but given a particular process these

results show what is involved in calculating the expectations of the moments.

Section 4

M. R. Leadbetter, On crossings of levels and curves by a wide class of stochastic processes, *Ann. Math. Statist.*, Vol. 37, pp. 260-267, 1966.

"In this paper, upcrossings, downcrossings and tangencies to levels and curves are discussed within a general framework. The mean number of crossings of a level (or curve) is calculated for a wide class of processes and it is shown that tangencies have probability zero in these cases. This extends results of Ito and Ylvisaker for stationary normal processes, to non-stationary and non-normal cases. In particular, the corresponding result given by Leadbetter and Cryer for normal, non-stationary can be slightly improved to apply under minimal conditions. An application is also given for an important non-normal process", (namely the envelope of a stationary normal process). (Author's introduction, *Annals of Mathematical Statistics*, Vol. 37, 1966.)

The biggest value of this paper may not be its theorems so much as the intrinsic value of their proofs and the discussions. It is shown just how far one may proceed with virtually no regularity conditions in the most general case; and specific results are obtained as corollaries. Among other important results it is shown that under very broad considerations there will, with probability 1, be no tangencies. This has a unifying effect on the previous papers mentioned in that depending on the method used to count crossings these prior formulas for the expected number of crossings may or may not have counted tangencies.

As an application Leadbetter considers the "envelope" $R(t)$ of a stationary normal process as defined by Rice (Section 3.7-3.9) and others, and proves that the expected number of crossings of a level u is equal to $e^{-u^2/2}$ times the expected number of crossings of the level $u = 0$. (This latter expectation is also computed.) It is mentioned that the expected number of

upcrossings and the expected number of downcrossings are each exactly one-half the above quantity.

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